A regression model with a hidden logistic process for feature extraction from time series

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Context: Feature extraction from the switch operation signals

- Signals of the consumed electrical power during switch operations

- Each switch operation consists of 5 successive electromechanical motions
- The signals present smooth or abrupt changes between different regimes
- The proposed solution: use an adapted regression model whose parameters will be used as the feature vector for each signal.
The piecewise regression approach

Piecewise polynomial regression model [McGee & Carleton 70]

- The data: \( \{(x_1, t_1), \ldots, (x_n, t_n)\} \)
  - \( x_i \): real dependant variable: the observation of the signal
  - \( t_i \): independant variable representing the time

- The piecewise polynomial regression model generating the signal \( x \) is:
  \[
  \forall i = 1, \ldots, n, \quad x_i = \beta_k^T r_i + \sigma_k \epsilon_i \quad ; \quad \epsilon_i \sim \mathcal{N}(0, 1)
  \]
  - \( k \) satisfies \( i \in I_k = (\gamma_k, \gamma_{k+1}] \): indexes of elements in segment \( k \)
  - \( r_i = (1, t_i, \ldots, t_i^p)^T \): time-dependant covariate vector in \( \mathbb{R}^{p+1} \)
  - \( \beta_k \): regression coefficients vector \( \in \mathbb{R}^{(p+1)} \) for the \( k^{th} \) segment

The model parameters

\((\psi, \gamma)\) with \( \psi = (\beta_1, \ldots, \beta_K, \sigma_1^2, \ldots, \sigma_K^2) \) and \( \gamma = (\gamma_1, \ldots, \gamma_{K+1}) \).
Parameter estimation for the piecewise regression model

Maximize the likelihood of $(\psi, \gamma)$ or equivalently minimize, with respect to $(\psi, \gamma)$:

$$J(\psi, \gamma) = \sum_{k=1}^{K} \sum_{i \in I_k} \left[ \log \sigma_k^2 + \frac{(x_i - \beta_k^T r_i)^2}{\sigma_k^2} \right].$$

- Global optimization using Fisher’s algorithm [Fisher 58] based on dynamic programming [Bellman 61; Lechevallier 90] since the criterion $J$ is additive on $k$
- Local optimization using an iterative variant of Fisher’s Algorithm [Samé et al. 07]

Time series approximation and segmentation

- $\hat{x}_i = \sum_{k=1}^{K} \hat{z}_{ik} \hat{\beta}_k^T r_i$ ; $\forall i = 1, \ldots, n$
- $\hat{z}_{ik} = 1$ if $i \in (\hat{\gamma}_k, \hat{\gamma}_{k+1}]$ ($x_i$ belongs to the $k^{th}$ segment) and $\hat{z}_{ik} = 0$ otherwise

- Using dynamic programming can be computationally expensive
- Provides a hard partition $\Rightarrow$ adapted for regimes with abrupt changes
The proposed regression based on hidden process approach

The global regression model

\[ \forall i = 1, \ldots, n, \quad x_i = \beta_{z_i}^T r_i + \sigma_{z_i} \epsilon_i \quad ; \quad \epsilon_i \sim N(0, 1), \]

- \( z_i \in \{1, \ldots, K\} \) hidden variable: the class label of the regression model generating \( x_i \)

\[ z = (z_1, \ldots, z_n) \] is a hidden discrete process

\( z_i \sim \mathcal{M}(1, \pi_{i1}(w), \ldots, \pi_{iK}(w)) \); where

\[ \pi_{ik}(w) = p(z_i = k; w) = \frac{\exp(w_k^T v_i)}{\sum_{\ell=1}^{K} \exp(w_{\ell}^T v_i)}, \]

- \( v_i = (1, t_i, \ldots, t_i^q)^T \) time-dependant covariate vector \( \in \mathbb{R}^{q+1} \)
- \( w = (w_1, \ldots, w_K) \) the parameter vector for the \( K \) \( k \) logistic functions \( \in \mathbb{R}^{K \times (q+1)} \)
Flexibility of the logistic transformation: Example for $K = 2$:

1. $\pi_{ik}(w)$ in relation to the dimension $q$ of $w_k$:

$q = 0$

$q = 1$

$q = 2$

$\Rightarrow q = 1$ guarantees segmentation into contiguous segments

2. $\pi_{ik}(w)$ in relation to $w_k$ for $q = 1$; we parametrize $w_k$ by $w_k = \lambda_k(\alpha_k, 1)^T$

$\alpha_1 = -2$

$\lambda_1 = -5$

$\Rightarrow$ The parameter $\lambda_k$ controls the quality of transitions between classes

$\Rightarrow$ The parameter $\alpha_k$ controls the transition time point.
Parameter estimation by maximum likelihood

- Derived mixture density

\[ p(x_i; \theta) = \sum_{k=1}^{K} \pi_{ik}(w) \mathcal{N}(x_i; \beta_k^T r_i, \sigma_k^2) \]

- Model parameters

\[ \theta = (w, \beta_1, \ldots, \beta_K, \sigma_1^2, \ldots, \sigma_K^2) \]

- Log-likelihood of \( \theta \):

\[ L(\theta; x) = \sum_{i=1}^{n} \log \sum_{k=1}^{K} \pi_{ik}(w) \mathcal{N}(x_i; \beta_k^T r_i, \sigma_k^2). \]

- Maximization of \( L(\theta; x) \) by a dedicated Expectation-Maximization (EM) algorithm [Dempster et al. 77].
Dedicated EM algorithm

Initialization: $\theta^{(0)}$

Repeat until convergence:

1. **E step: Expectation** (at iteration $m$)
   
   Compute the conditional expectation of the complete log-likelihood $L(\theta; x, z)$
   
   $$Q(\theta, \theta^{(m)}) = E \left[ L(\theta; x, z) | x, \theta^{(m)} \right]$$
   
   $$= \sum_{i=1}^{n} \sum_{k=1}^{K} \tau_{ik}^{(m)} \log \pi_{ik}(w) + \sum_{i=1}^{n} \sum_{k=1}^{K} \tau_{ik}^{(m)} \log \mathcal{N}(x_i; \beta_k^T r_i, \sigma_k^2),$$

   $$Q_1(w)$$

   $$Q_2(\beta_k, \sigma_k^2; k=1,\ldots,K)$$

2. **M step: Maximization** (at iteration $m$)

   Compute $\theta^{(m+1)} = \arg \max_{\theta} Q(\theta, \theta^{(m)})$
Details of the M step

1. Maximization of $Q_2$ with respect to $\beta_k$ s: Analytic solutions of $K$ separate polynomial regression problems weighted by the $\tau_{ik}^{(m)}$ s:

   • $\beta_k^{(m+1)} = (M^T \Gamma_k^{(m)} M)^{-1} M^T \Gamma_k^{(m)} x$

   where $M$ is the design matrix and $\Gamma_k^{(m)} = \text{diag}(\tau_{1k}^{(m)}, \ldots, \tau_{nk}^{(m)})$.

   Maximization of $Q_2$ with respect to $\sigma_k^2$ s

   • $\sigma_k^{2(m+1)} = \frac{1}{\sum_{i=1}^{n} \tau_{ik}^{(m)}} \sum_{i=1}^{n} \tau_{ik}^{(m)} (x_i - \beta_k^{T(m+1)} r_i)^2$.

2. Maximize $Q_1$ with respect to $w$: Solve a multiclass convex logistic regression problem weighted by the $\tau_{ik}^{(m)}$ s $\Rightarrow$ IRLS algorithm [Chen 99, Green 84, Krishnapuram 05]

   $$w^{(c+1)} = w^{(c)} - \left[ \frac{\partial^2 Q_1(w)}{\partial w \partial w^T} \right]^{-1} \frac{\partial Q_1(w)}{\partial w} \bigg|_{w=w^{(c)}}$$

   $\Rightarrow$ Applying the IRLS algorithm provides the parameter $w^{(m+1)}$. 

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Time series approximation and segmentation with the proposed model

Time series approximation

- As in standard regression, given the estimated parameters, $x_i$ is approximated by its expectation $\forall i = 1, \ldots, n$:

$$\hat{x}_i = E(x_i; \hat{\theta}) = \int_{\mathbb{R}} x_i p(x_i; \hat{\theta}) dx_i = \sum_{k=1}^{K} \pi_{ik}(\hat{w}) \hat{\beta}_k^T r_i.$$

A sum of polynomials weighted by the logistic probabilities $\pi_{ik}(\hat{w})$’s.

$\Rightarrow$ Adapted for a smooth or abrupt transitions between the regression models.

Time series segmentation

- The estimated class label $\hat{z}_i$ of $x_i$ is given by the rule:

$$\hat{z}_i = \arg \max_{1 \leq k \leq K} \pi_{ik}(\hat{w}).$$
Experiments using simulated data

- Evaluation criteria:
  1. Misclassification error rate (segmentation error)
  2. Error between the true simulated curve without noise and the estimated curve (Denoising error):

\[
\frac{1}{n} \sum_{i=1}^{n} (E(x_i; \theta) - E(x_i; \hat{\theta}))^2
\]

- Comparison with the two piecewise regression approaches

- 2 situations of signals with
  - \( K = 3, \ p = 2, \ q = 1 \)
  - Varying the sample size \( n = 100, 200, \ldots, 1000 \)
  - \( \sigma_1^2 = 4, \ \sigma_2^2 = 10, \ \sigma_3^2 = 15 \)

- Assessment criteria are averaged over 20 samples for each value of \( n \)
Example of simulated signals

Situation 1

Situation 2
Results 1 (Situation 1)

Experiments

- Denoising error
  - Proposed algorithm
  - Fisher's algorithm
  - Iterative algorithm

- Misclassification error rate
  - Proposed algorithm
  - Fisher's algorithm
  - Iterative algorithm

Sample size (n)

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Results 2 (Situation 2)

Denoising error

Sample size (n)

Misclassification error rate

Sample size (n)
Computing time

Average running time

- Proposed algorithm
- Fisher’s algorithm
- Iterative algorithm
Application to real signals

Original signal and estimated signal

Probabilities of the regression models

Corresponding regression models
Conclusion

- In contrast with the basic polynomial regression, the proposed approach authorizes the regression parameters to vary over time ⇒ Accurate modeling of nonlinear signals

- The proposed model integrates a logistic process which makes possible to change smoothly within various possible regression models

- In addition to feature extraction, this approach can be used to denoise and segment time series (or signals)

- Computationally efficient.
Thank you!